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ECS315 2016/1 Part V.1 Dr.Prapun

11 Multiple Random Variables

One is often interested not only in individual random variables, but also in relationships between two or more random variables. Furthermore, one often wishes to make inferences about one random variable on the basis of observations of other random variables.

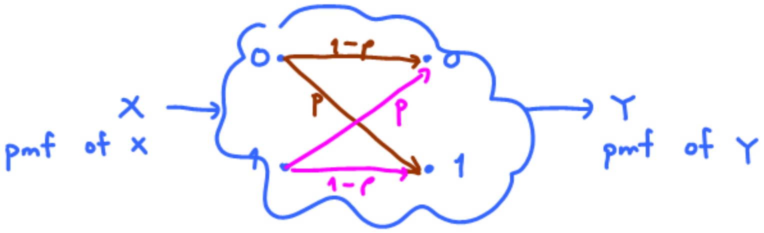
Example 11.1. If the experiment is the testing of a new medicine, the researcher might be interested in cholesterol level, blood pressure, and the glucose level of a test person.

11.1 A Pair of Discrete Random Variables

In this section, we consider two discrete random variables, say X and Y, simultaneously.

11.2. The analysis are different from Section 9.2 in two main aspects. First, there may be *no* deterministic relationship (such as $Y = g(X)$) between the two random variables. Second, we want to look at both random variables as a whole, not just X alone or Y alone.

Example 11.3. Communication engineers may be interested in the input X and output Y of a communication channel. BSC



Example 11.4. Of course, to rigorously define (any) random variables, we need to go back to the sample space Ω. Recall Example 7.7 where we considered several random variables defined on the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ where the outcomes are equally likely. In that example, we define $X(\omega) = \omega$ and $Y(\omega) = (\omega - 3)^2$.

Example 11.5. Consider the scores of 20 students below:

$$\underbrace{10, 9, 10, 9, 9, 10, 9, 10, 10, 9}_{\text{Room \#1}}, \underbrace{1, 3, 4, 6, 5, 5, 3, 3, 1, 3}_{\text{Room \#2}}.$$

The first ten scores are from (ten) students in room #1. The last 10 scores are from (ten) students in room #2.

Suppose we have the a score report card for each student. Then, in total, we have 20 report cards.

Figure 31: In Example 11.5, we pick a report card randomly from a pile of cards.

I pick one report card up randomly. Let X be the score on that card.

- What is the chance that $X = 10$? (Ans: $p_X(10) = P[X = 10] = 5/20 = 1/4$.)

- What is the chance that $X > 5$? (Ans: $P[X > 5] = 11/20$.)

Now, let the random variable Y denote the room# of the student whose report card is picked up.

- What is the probability that $X = 10$ and $Y = 2$?

- What is the probability that $X = 10$ and $Y = 1$?

- What is the probability that $X > 5$ and $Y = 1$?

- What is the probability that $X > 5$ and $Y = 2$?

Now suppose someone informs me that the report card which I picked up is from a student in room #1. (He may be able to tell this by the color of the report card of which I have no knowledge.) I now have an extra information that $Y = 1$.

- What is the probability that $X > 5$ given that $Y = 1$?

- What is the probability that $X = 10$ given that $Y = 1$?

11.6. Recall that, in probability, “,” means “and”. For example,

$$P[X = x, Y = y] = P[X = x \text{ and } Y = y]$$

and

$$\begin{aligned} P[3 \le X < 4, Y < 1] &= P[3 \le X < 4 \text{ and } Y < 1] \\ &= P[X \in [3,4) \text{ and } Y \in (-\infty, 1)]. \end{aligned}$$

In general, the event

$$[\text{“Some condition(s) on } X\text{”}, \text{“Some condition(s) on } Y\text{”}]$$

is the same as the intersection of two events:

$$[\text{“Some condition(s) on } X\text{”}] \cap [\text{“Some condition(s) on } Y\text{”}]$$

which simply means both statements happen.

More technically,

$$[X \in B, Y \in C] = [X \in B \text{ and } Y \in C] = [X \in B] \cap [Y \in C]$$

and

$$\begin{aligned} P[X \in B, Y \in C] &= P[X \in B \text{ and } Y \in C] \\ &= P([X \in B] \cap [Y \in C]). \end{aligned}$$

Remark: Linking back to the original sample space, this shorthand actually says

$$\begin{aligned} [X \in B, Y \in C] &= [X \in B \text{ and } Y \in C] \\ &= \{\omega \in \Omega : X(\omega) \in B \text{ and } Y(\omega) \in C\} \\ &= \{\omega \in \Omega : X(\omega) \in B\ \} \cap \{\omega \in \Omega : Y(\omega) \in C\} \\ &= [X \in B] \cap [Y \in C]. \end{aligned}$$

11.7. The concept of conditional probability can be straightforwardly applied to discrete random variables. For example,

$$P\left[\text{“Some condition(s) on } X\text{”} \mid \text{“Some condition(s) on } Y\text{”}\right] \quad (26)$$

is the conditional probability $P(A|B)$ where

$$\begin{aligned} A &= [\text{“Some condition(s) on } X\text{”}] \text{ and} \\ B &= [\text{“Some condition(s) on } Y\text{”}]. \end{aligned}$$

Recall that $P(A|B) = P(A \cap B)/P(B)$. Therefore,

$$P\left[X = x | \, Y = y\right] = \frac{P\left[X = x \text{ and } Y = y\right]}{P\left[Y = y\right]},$$

and

$$P\left[3 \leq X < 4 | \, Y < 1\right] = \frac{P\left[3 \leq X < 4 \text{ and } Y < 1\right]}{P\left[Y < 1\right]}$$

More generally, (26) is

$$\begin{aligned} &= \frac{P\left([\text{“Some condition(s) on } X\text{”}] \cap [\text{“Some condition(s) on } Y\text{”}]\right)}{P\left([\text{“Some condition(s) on } Y\text{”}]\right)} \\ &= \frac{P\left([\text{“Some condition(s) on } X\text{”}, \text{“Some condition(s) on } Y\text{”}]\right)}{P\left([\text{“Some condition(s) on } Y\text{”}]\right)} \\ &= \frac{P[\text{“Some condition(s) on } X\text{”}, \text{“Some condition(s) on } Y\text{”}]}{P[\text{“Some condition(s) on } Y\text{”}]} \end{aligned}$$

More technically,

$$\begin{aligned} P\left[X \in B | \, Y \in C\right] = P\left([X \in B] \, | [Y \in C]\right) &= \frac{P\left([X \in B] \cap [Y \in C]\right)}{P\left([Y \in C]\right)} \\ &= \frac{P\left[X \in B, Y \in C\right]}{P\left[Y \in C\right]}. \end{aligned}$$

Definition 11.8. ***Joint pmf***: If X and Y are two discrete random variables (defined on a same sample space with probability measure P), the function $p_{X,Y}(x,y)$ defined by

$$p_{X,Y}(x,y) = P[X = x, Y = y]$$

and

Note that $\sum_{(x,y)} p_{X,Y}(x,y) = 1$

all possible pairs (x,y)

is called the ***joint probability mass function*** of X and Y.

(a) We can visualize the joint pmf via stem plot. See Figure 32.

(b) To evaluate the probability for a statement that involves both X and Y random variables:

$P[\text{condition(s) on } X \text{ and/or } Y]$

We first find ① all pairs (x,y) that satisfy ② the condition(s) in the statement, and then ③ add up all the corresponding values from the joint pmf .

More technically, we can then evaluate $P[(X,Y) \in R]$ by

$$P[(X,Y) \in R] = \sum_{(x,y):(x,y)\in R} p_{X,Y}(x,y).$$

Example 11.9 (F2011)**.** Consider random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} c(x+y), & x \in \{1,3\} \text{ and } y \in \{2,4\}, \\ 0, & \text{otherwise.} \end{cases}$$

Four possible cases

x^2+y^2	(x,y)	$p_{X,Y}(x,y)$
5	(1,2)	$3c = 3/20$
17	(1,4)	$5c = 5/20$
13	(3,2)	$5c = 5/20$
25	(3,4)	$7c = 7/20$

(a) Check that $c = 1/20$.

$\sum_{(x,y)} p_{X,Y}(x,y) = 1 \Rightarrow 20c = 1$

$\Rightarrow c = 1/20$

(b) Find $P[X^2 + Y^2 = 13]$.

$= 5c = \frac{5}{20} = \frac{1}{4}$

(c) $P[X^2+Y^2 < 20] = 3c + 5c + 5c = 13c$

$= 13/20$

$$\mathbf{P}_{X,Y} = \begin{array}{c|cc} x \backslash y & 2 & 4 \\ \hline 1 & 3c & 5c \\ 3 & 5c & 7c \end{array}$$

In most situation, it is much more convenient to focus on the "important" part of the joint pmf. To do this, we usually present the joint pmf (and the conditional pmf) in their matrix forms:

x^2+y^2

$$\begin{array}{cc|cc} & y^2 & 4 & 16 \\ x^2 & x \backslash y & 2 & 4 \\ \hline 1 & 1 & 5 & 17 \\ 9 & 3 & 13 & 25 \end{array}$$

Definition 11.10. When both X and Y take finitely many values (both have finite supports), say $S_X = \{x_1, \ldots, x_m\}$ and $S_Y = \{y_1, \ldots, y_n\}$, respectively, we can arrange the probabilities $p_{X,Y}(x_i, y_j)$ in an $m \times n$ matrix

$$\begin{array}{c|cccc} x \backslash Y & y_1 & y_2 & \cdots & y_n \\ \hline x_1 & p_{X,Y}(x_1,y_1) & p_{X,Y}(x_1,y_2) & \ldots & p_{X,Y}(x_1,y_n) \\ x_2 & p_{X,Y}(x_2,y_1) & p_{X,Y}(x_2,y_2) & \ldots & p_{X,Y}(x_2,y_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_m & p_{X,Y}(x_m,y_1) & p_{X,Y}(x_m,y_2) & \ldots & p_{X,Y}(x_m,y_n) \end{array}. \tag{27}$$

- We shall call this matrix the ***joint pmf matrix***.
- The sum of all the entries in the matrix is one.

$$\sum_{(x,y)} p_{X,Y}(x,y) = 1$$

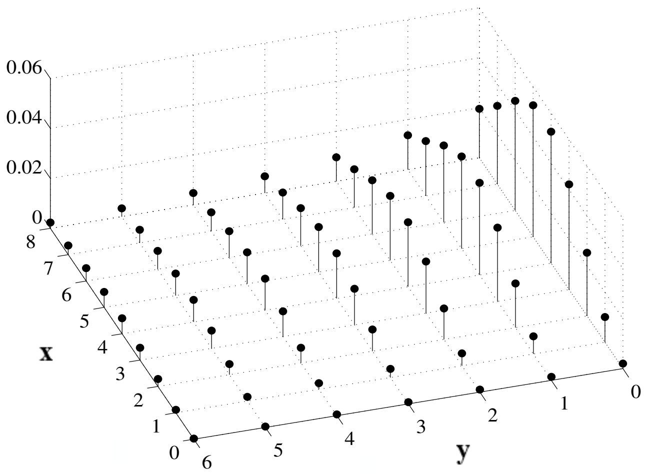


Figure 32: Example of the plot of a joint pmf. [9, Fig. 2.8]

- $p_{X,Y}(x, y) = 0$ if[48] $x \notin S_X$ or $y \notin S_Y$. In other words, we don't have to consider the x and y outside the supports of X and Y, respectively.

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[48] To see this, note that $p_{X,Y}(x, y)$ can not exceed $p_X(x)$ because $P(A \cap B) \leq P(A)$. Now, suppose at $x = a$, we have $p_X(a) = 0$. Then $p_{X,Y}(a, y)$ must also $= 0$ for any y because it can not exceed $p_X(a) = 0$. Similarly, suppose at $y = a$, we have $p_Y(a) = 0$. Then $p_{X,Y}(x, a) = 0$ for any x.

11.11. From the joint pmf, we can find $p_X(x)$ and $p_Y(y)$ by

$$p_X(x) = \sum_y p_{X,Y}(x,y) \tag{28}$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y) \tag{29}$$

In this setting, $p_X(x)$ and $p_Y(y)$ are call the ***marginal pmfs*** (to distinguish them from the joint one).

(a) Suppose we have the joint pmf matrix in (27). Then, the sum of the entries in the ith row is[49] $p_X(x_i)$, and
the sum of the entries in the jth column is $p_Y(y_j)$:

$$p_X(x_i) = \sum_{j=1}^{n} p_{X,Y}(x_i,y_j) \quad \text{and} \quad p_Y(y_j) = \sum_{i=1}^{m} p_{X,Y}(x_i,y_j)$$

(b) In `MATLAB`, suppose we save the joint pmf matrix as `P_XY`, then the marginal pmf (row) vectors `p_X` and `p_Y` can be found by

```
p_X = (sum(P_XY,2))'
p_Y = (sum(P_XY,1))
```

Example 11.12. Consider the following joint pmf matrix

(sum along each row)

$$\begin{array}{c|cccc|l} x\backslash y & 0 & 1 & 2 & 3 & \\ \hline 0 & 0.1 & 0 & 0.2 & 0 & \xrightarrow{\Sigma} 0.3 \\ 1 & 0 & 0.5 & 0 & 0 & \xrightarrow{\Sigma} 0.5 \\ 2 & 0 & 0.1 & 0.1 & 0 & \xrightarrow{\Sigma} 0.2 \\ \hline & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \Sigma\downarrow & \\ & 0.1 & 0.6 & 0.3 & 0 & \end{array}$$

$$p_X(x) = \begin{cases} 0.3, & x=0, \\ 0.5, & x=1, \\ 0.2, & x=2, \\ 0, & \text{otherwise.} \end{cases}$$

$$p_Y(y) = \begin{cases} 0.1, & y=0, \\ 0.6, & y=1, \\ 0.3, & y=2, \\ 0, & \text{otherwise} \end{cases}$$

① $p_{X,Y}(1,1) = 0.5 = P[X=1 \text{ and } Y=1]$

$P[X=0 \text{ and } Y=2] = 0.2$

② $P[XY>2] = 0.1+0+0 = 0.1$

$$xy \quad \begin{array}{c|cccc} x\backslash y & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 4 & 6 \end{array}$$

③ $p_X(2) = P[X=2] = 0+0.1+0.1+0$
$= 0.2$

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[49] To see this, we consider $A = [X = x_i]$ and a collection defined by $B_j = [Y = y_j]$ and $B_0 = [Y \notin S_Y]$. Note that the collection $B_0, B_1, \ldots, B_n$ partitions Ω. So, $P(A) = \sum_{j=0}^{n} P(A \cap B_j)$. Of course, because the support of Y is S_Y, we have $P(A \cap B_0) = 0$. Hence, the sum can start at $j = 1$ instead of $j = 0$.

Ex. $P_{X|Y}(0|1) = P[X=0|Y=1]$

Definition 11.13. The ***conditional pmf*** of X given Y is defined as

$$p_{X|Y}(x|y) = P[\underbrace{X=x}_{A}|\underbrace{Y=y}_{B}] = P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

which gives

$P(A\cap B) = P(A|B)P(B) = P(B|A)P(A)$

$$\longrightarrow p_{X,Y}(x,y) = p_{X|Y}(x|y)p_Y(y) = p_{Y|X}(y|x)p_X(x). \tag{30}$$

11.14. Equation (30) is quite important in practice. In most cases, systems are naturally defined/given/studied in terms of their conditional probabilities, say $p_{Y|X}(y|x)$. Therefore, it is important the we know how to construct the joint pmf from the conditional pmf.

Example 11.15. Consider a binary symmetric channel. Suppose the input X to the channel is Bernoulli(0.3). At the output Y of this channel, the crossover (bit-flipped) probability is 0.1. Find the joint pmf $p_{X,Y}(x,y)$ of X and Y.

p

$P_{Y|X}(1|0) = 0.1$

$P_{Y|X}(0|1) = 0.1$

$P_{Y|X}(1|1) = 0.9$

$P_{Y|X}(0|0) = 0.9$

$P_{X,Y}(x,y) = P_{Y|X}(y|x)\, p_X(x)$

$$p_X(x) = \begin{cases} 0.3, & x=1, \\ 0.7, & x=0, \\ 0, & \text{otherwise.} \end{cases}$$

$$P_{Y|X}\ \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 0.9 & 0.1 \\ 1 & 0.1 & 0.9 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.7} \\[1em] \xrightarrow[\times 0.3]{} \end{array} \quad \begin{array}{c|cc} x\backslash y & 0 & 1 \\ \hline 0 & 0.63 & 0.07 \\ 1 & 0.03 & 0.27 \\ & \Sigma\downarrow & \Sigma\downarrow \\ & 0.66 & 0.34 \end{array} = P_{X,Y}$$

$$P_Y(y) = \begin{cases} 0.66, & y=0, \\ 0.34, & y=1, \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 11.16. Toss-and-Roll Game:

Step 1 Toss a fair coin. Define X by

$$X = \begin{cases} 1, & \text{if result} = \text{H}, \\ 0, & \text{if result} = \text{T}. \end{cases}$$

Step 2 You have two dice, Dice 1 and Dice 2. Dice 1 is fair. Dice 2 is unfair with $p(1) = p(2) = p(3) = \frac{2}{9}$ and $p(4) = p(5) = p(6) = \frac{1}{9}$.

(i) If $X = 0$, roll Dice 1.

(ii) If $X = 1$, roll Dice 2.

Record the result as Y.

Find the joint pmf $p_{X,Y}(x, y)$ of X and Y.

Exercise 11.17 (F2011)**.** Continue from Example 11.9. Random variables X and Y have the following joint pmf

$$p_{X,Y}(x, y) = \begin{cases} c(x+y), & x \in \{1,3\} \text{ and } y \in \{2,4\}, \\ 0, & \text{otherwise.} \end{cases}$$

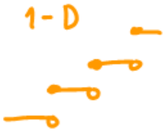
(a) Find $p_X(x)$.

(b) Find $\mathbb{E}X$.

(c) Find $p_{Y|X}(y|1)$. Note that your answer should be of the form

$$p_{Y|X}(y|1) = \begin{cases} ?, & y = 2, \\ ?, & y = 4, \\ 0, & \text{otherwise.} \end{cases}$$

(d) Find $p_{Y|X}(y|3)$.



Definition 11.18. The ***joint cdf*** of X and Y is defined by

$$F_{X,Y}(x, y) = P[X \le x, Y \le y].$$

Definition 11.19. Two random variables X and Y are said to be ***identically distributed*** if, for every B, $P[X \in B] = P[Y \in B]$.

Example 11.20. Roll a dice twice. Let X be the result from the first roll. Let Y be the result from the second roll.

- X and Y are not the same. (Most of the time, they will be different. By chance, they occasionally take the same value.)
-

$$P[X > 3] = P[Y > 3]$$

$$P[|X-5| = 1] = P[|Y-5| = 1]$$

In words, for any probability statement about X (and only X) if we replace X by Y, we get the same probability.

x	$p_X(x)$	$z = 1 - x$
0	1/2	1
1	1/2	0

$$p_Z(z) = \begin{cases} 1/2, & z = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 11.21. Let $X \sim$ Bernoulli$(1/2)$. Let $Y = X$ and $Z = 1 - X$. Then, all of these random variables are identically distributed.

11.22. The following statements are equivalent:

(a) Random variables X and Y are ***identically distributed***.

Any statement defined using only X

(b) For every B, $P[X \in B] = P[Y \in B]$

the same statement but replace X by Y

(c) $p_X(c) = p_Y(c)$ for all c

(d) $F_X(c) = F_Y(c)$ for all c

Definition 11.23. Two random variables X and Y are said to be ***independent*** if the events $[X \in B]$ and $[Y \in C]$ are independent for all sets B and C.

Review: Events A and B $(A \perp\!\!\!\perp B)$ are independent iff $P(A \cap B) = P(A)P(B)$

11.24. The following statements are equivalent:

(a) Random variables X and Y are ***independent***.

very powerful easy to apply but difficult to show

(b) $[X \in B] \perp\!\!\!\perp [Y \in C]$ for all B, C.

(c) $P[X \in B, Y \in C] = P[X \in B] \times P[Y \in C]$ for all B, C.

(d) $p_{X,Y}(x, y) = p_X(x) \times p_Y(y)$ for all x, y.

(e) $F_{X,Y}(x, y) = F_X(x) \times F_Y(y)$ for all x, y.

If $X \perp\!\!\!\perp Y$, $P[X > 7, Y < 3] = P[X > 7] P[Y < 3]$

$P[X = x, Y = y] = P[X = x] P[Y = y]$

Definition 11.25. Two random variables X and Y are said to be ***independent and identically distributed (i.i.d.)*** if X and Y are both independent and identically distributed.

11.26. Being identically distributed does not imply independence. Similarly, being independent, does not imply being identically distributed.

Example 11.27. Roll a dice. Let X be the result. Set $Y = X$.

① X, Y i.d.? $p_Y(c) = p_X(c)$ $\forall c$?

$$p_Y(c) = P[Y=c] = P[X=c] = p_X(c) \text{ for any } c$$

Therefor, X and Y are identically distributed.

② $p_{X,Y}(1,1) \equiv P[X=1, Y=1] = P[Y=1|X=1]\,P[X=1] = 1 \times \frac{1}{6} = 1/6$

$p_X(1) = \frac{1}{6} = p_Y(1) \leftarrow$ i.d.

$p_{X,Y}(1,1) \neq p_X(1)\,p_Y(1)$ $\Rightarrow X \not\perp\!\!\!\perp Y$

Example 11.28. Suppose the pmf of a random variable X is given by

$$p_X(x) = \begin{cases} 1/4, & x = 3, \\ \cancel{\alpha}, 3/4 & x = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Let Y be another random variable. Assume that X and Y are i.i.d.

Find

(a) α, $\sum_x p_X(x) = 1 \Rightarrow \frac{1}{4} + \alpha = 1 \Rightarrow \alpha = \frac{3}{4}$

(b) the pmf of Y, and

(c) the joint pmf of X and Y.

(b) Because X and Y are identically distributed, we have $p_Y(c) = p_X(c)$ for any c.

In particular,

$$p_Y(y) = p_X(y) = \begin{cases} 1/4, & y = 3, \\ 3/4, & y = 4, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Because X and Y are independent,

$$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$$

$$\begin{array}{c|cc} x\backslash y & 3 & 4 \\ \hline 3 & \frac{1}{4}\times\frac{1}{4} & \frac{1}{4}\times\frac{3}{4} \\ 4 & \frac{3}{4}\times\frac{1}{4} & \frac{3}{4}\times\frac{3}{4} \end{array}$$

$$= \begin{array}{c|cc} x\backslash y & 3 & 4 \\ \hline 3 & 1/16 & 3/16 \\ 4 & 3/16 & 9/16 \end{array}$$

$$p_{X,Y}(x,y) = \begin{cases} 1/16, & (x,y) = (3,3), \\ 3/16, & (x,y) = (3,4) \text{ or } (4,3), \\ 9/16, & (x,y) = (4,4), \\ 0, & \text{otherwise.} \end{cases}$$

Example 11.29. Consider a pair of random variables X and Y whose joint pmf is given by

$$\sum_{(x,y)} p_{X,Y}(x,y) = 1$$

$$\frac{1+2+4}{15} + \beta = 1$$

$$\beta = 8/15$$

$$p_{X,Y}(x,y) = \begin{cases} 1/15, & x = 3, y = 1, \\ 2/15, & x = 4, y = 1, \\ 4/15, & x = 3, y = 3, \\ \cancel{\beta,}\ 8/15 & x = 4, y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y identically distributed? No

(b) Are X and Y independent? Yes

(a)

$$\begin{array}{c|cc|c} x \backslash y & 1 & 3 & \\ \hline 3 & 1/15 & 4/15 & \xrightarrow{\Sigma} 1/3 \\ 4 & 2/15 & 8/15 & \xrightarrow{\Sigma} 2/3 \\ \hline & \downarrow \Sigma & \downarrow \Sigma & \\ & 1/5 & 4/5 & \end{array}$$

$$p_X(x) = \begin{cases} 1/3, & x = 3, \\ 2/3, & x = 4, \\ 0, & \text{otherwise.} \end{cases}$$

$$p_Y(y) = \begin{cases} 1/5, & y = 1, \\ 4/5, & y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

So, X and Y are not identically distributed.
(For example, consider $c = 3$;

$$p_X(c) = \frac{1}{3} \neq \frac{1}{5} = p_Y(c).)$$

(b) $p_{X,Y}(x,y) \overset{?}{=} p_X(x)\, p_Y(y)$

$$p_X(x) \quad \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 3 & \frac{1}{3} \times \frac{1}{5} & \frac{1}{3} \times \frac{4}{5} \\ 4 & \frac{2}{3} \times \frac{1}{5} & \frac{2}{3} \times \frac{4}{5} \end{array}$$

Because

$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$

for all x, y,

we conclude that

$X \perp\!\!\!\perp Y$.

Back to discrete RVs.

One RV

pmf : $p_X(x) = P[X=x]$

Two RVs

joint pmf : $p_{X,Y}(x,y) = P[X=x \overset{\text{"and"}}{,} Y=y]$.

summarized by a joint pmf matrix

$$\mathbf{P}_{X,Y} = \begin{array}{l} {}^{x \setminus y} \\ \left[\qquad\qquad \right] \end{array}$$

marginal pmfs

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$ ← sum along the corresponding row of $\mathbf{P}_{X,Y}$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$ ← sum along the corresponding column of $\mathbf{P}_{X,Y}$

identically distributed : $p_X(c) = p_Y(c)$ for all c

(same pmf)

independent : $p_{X,Y}(x,y) = p_X(x) \underset{\uparrow \atop \text{multiplication}}{\times} p_Y(y)$

$\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

n

11.2 Extending the Definitions to Multiple RVs

Definition 11.30. Joint pmf:

$$p_{\vec{X}}(\vec{x}) = p_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = P\left[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n\right].$$

Joint cdf:

$$F_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = P\left[X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n\right].$$

11.31. Marginal pmf:

$$p_X(x) = \underbrace{\sum_y \sum_z}_{\text{sum over all possible } y,z} p_{X,Y,Z}(x,y,z)$$

Ex.

x	y	z	$p_{X,Y,Z}(x,y,z)$
0	1	0	1/7
0	0	1	2/7
0	1	1	3/7
1	1	1	1/7

$p_X(0) = \frac{1+2+3}{7} = \frac{6}{7}$

$p_X(1) = 1/7$

Definition 11.32. ***Identically distributed*** random variables: The following statements are equivalent.

(a) Random variables $X_1, X_2, \dots$ are ***identically distributed***

(b) For every B, $P\left[X_j \in B\right]$ does not depend on j.

(c) $p_{X_i}(c) = p_{X_j}(c)$ for all c, i, j.

(d) $F_{X_i}(c) = F_{X_j}(c)$ for all c, i, j.

Definition 11.33. ***Independence*** among finite number of random variables: The following statements are equivalent.

(a) $X_1, X_2, \dots, X_n$ are ***independent***

(b) $[X_1 \in B_1], [X_2 \in B_2], \dots, [X_n \in B_n]$ are independent, for all $B_1, B_2, \dots, B_n$.

(c) $P\left[X_i \in B_i, \forall i\right] = \prod_{i=1}^n P\left[X_i \in B_i\right]$, for all $B_1, B_2, \dots, B_n$.

product (multiplication)

(d) $p_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = \prod_{i=1}^n p_{X_i}(x_i)$ for all $x_1, x_2, \dots, x_n$.

(e) $F_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = \prod_{i=1}^n F_{X_i}(x_i)$ for all $x_1, x_2, \dots, x_n$.

Example 11.34. Toss a coin n times. For the ith toss, let

$$X_i = \begin{cases} 1, & \text{if H happens on the } i\text{th toss,} \\ 0, & \text{if T happens on the } i\text{th toss.} \end{cases}$$

Each X_i is Bernoulli(p)

We then have a collection of i.i.d. random variables $X_1, X_2, X_3, \dots, X_n$.

$X_1 + X_2 + \dots + X_n \sim$ binomial(n,p)

$=$ #H in n trials

Example 11.35. Roll a dice n times. Let N_i be the result of the ith roll. We then have another collection of i.i.d. random variables $N_1, N_2, N_3, \ldots, N_n$.

Example 11.36. Let X_1 be the result of tossing a coin. Set $X_2 = X_3 = \cdots = X_n = X_1$.

Then, all the X_i's are Bernoulli(p) $\Rightarrow$ They are identically distributed.

However, they are not independent.

11.37. If $X_1, X_2, \ldots, X_n$ are independent, then so is any subcollection of them.

11.38. For i.i.d. $X_i \sim$ Bernoulli(p), $Y = X_1 + X_2 + \cdots + X_n$ is $\mathcal{B}(n, p)$.

binomial

Definition 11.39. A ***pairwise independent*** collection of random variables is a collection of random variables any two of which are independent.

(a) Any collection of (mutually) independent random variables is pairwise independent

(b) Some pairwise independent collections are not independent. See Example (11.40).

$$p_X(x) = \begin{cases} 1/2, & x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 11.40. Let suppose X, Y, and Z have the following joint probability distribution: $p_{X,Y,Z}(x, y, z) = \frac{1}{4}$ for $(x, y, z) \in \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$. This, for example, can be constructed by starting with independent X and Y that are Bernoulli-$\frac{1}{2}$. Then set $Z = X \oplus Y = X + Y \mod 2$.

$$\begin{cases} 1/4, & x, y \in \{0,1\} \\ 0, & \text{otherwise} \end{cases}$$
$$\| $$
$$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$$

$$p_{Y,Z}(y,z) = p_Y(y)\,p_Z(z)$$

$$p_{X,Z}(x,z) = p_X(x)\,p_Z(z)$$

(a) X, Y, Z are pairwise independent. ✓

(b) X, Y, Z are not independent.

x	y	z	$p_{X,Y,Z}(x,y,z)$
0	0	0	1/4
0	1	1	1/4
1	0	1	1/4
1	1	0	1/4

$$p_{X,Y,Z}(x,y,z) = p_X(x)\,p_Y(y)\,p_Z(z)$$

$$p_{X,Y,Z}(1,1,1) \overset{?}{=} p_X(1)\,p_Y(1)\,p_Z(1)$$

$$0 \neq \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$